PARTITION THEOREM IN POPULATIONS

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Abstract. Abstract. In this talk, I will state explain the new theorem that I have proved recently in formal demography. The conditions to identify the stationary status of a population are enough. The renewal process principles of stationary populations may not work when the sub-populations in large populations have non-steady equilibriums. The proof of the partition theorem will be provided.

Population partitions, stationary population and non-stationary population

Proposition. *Life Table Identity (SPI) partitions a randomly selected large population into stationary and non-stationary components.*

Proof. Stationary Population Identity (SPI) holds for a population, means, that in a life table the fraction of population at age x (say, $f_1(x)$) is equal to the fraction, $f_2(x)$ of the population who will live $x-$years [5, 4, 2, 6, 1, 3, 7], i.e. $f_1(x) = f_2(x)\ \forall\, x \in [0, \omega)$. Here ω is the maximum age in a life table.

Let $P(t)$ be the random population and $P_x(t)$ be the population at age x at a time t, where $P(t)$ can be expressed as $P(t) = \int_0^{\omega} P_x(t)dx$ or $P(t) = \Sigma_{x=0}^{\omega} P_x(t)$. We do not know whether $P(t)$ is a stationary or not. Stationary component of $P(t)$ (say, $M(t)$), we define as, a sub-collection of $P(t)$ who are aged y for $y \in [0, \omega)$ for which $f_1(y) = f_2(y)$ holds for every y, and non-stationary component of $P(t)$ (say, $N(t)$), we define as, a sub-collection of $P(t)$ whose age is z for $z \in [0, \omega)$ for which $f_1(z) \neq f_2(z)$. The sum of the sizes of $M(t)$ and $N(t)$ will be $P(t)$.

Let us choose all individuals at age y in $P(t)$ and denote the fraction, $f_1(y)$, as

$$f_1(y) = \frac{P_y(t)}{P(t)}. \tag{0.1}$$

From the life table constructed for the population $P(t)$,we can obtain the fraction of population who have expected remaining years within $[0, \omega)$,and compare the fraction of population at each age $x \in [0, \omega)$ in $P(t)$.That is, we will obtain $f_2(x)\ \forall x \in [0, \omega)$. Let,

$$f_2(y) = \frac{L_y(t)}{L(t)}, \tag{0.2}$$

where $L_y(t)$ is the life table population at time t whose remaining years to live is y and $L(t)$ is total life table population. We match $\frac{P_{y_1}(t)}{P(t)}$ for a given $y_1 \in [0, \omega)$ in (0.1) with $\frac{L_x(t)}{L(t)}$ for each

$x \in [0,\omega)$. If $\frac{P_{y_1}(t)}{P(t)}$ is equal to $\frac{L_y(t)}{L(t)}$ for some x,then we call the corresponding life table fraction of the population as $\frac{L_{y_1}(t)}{L(t)}$ $(say, f_2(y))$. This follows,

$$f_1(y_1) = \frac{P_{y_1}(t)}{P(t)} = \frac{L_{y_1}(t)}{L(t)} = f_2(y_1). \tag{0.3}$$

Suppose $\frac{P_{y_1}(t)}{P(t)}$ does not equal to fraction $\frac{L_x(t)}{L(t)}$ $\forall x \in [0,\omega)$, then we denote y_1 by z_1 and write this situation as $f_1(z_1) \neq f_2(z_1)$. We will continue matching $\frac{P_{y_2}(t)}{P(t)}$ for some $y_2 \neq y_1$ and $y_2 \in [0,\omega)$ with $\frac{L_x(t)}{L(t)}$ for each $x \in [0,\omega)$ except for $x = y_1$. If there is a value of $\frac{L_x(t)}{L(t)}$ that equals $\frac{P_{y_2}(t)}{P(t)}$, we call the corresponding fraction in life table population as $\frac{L_{y_2}(t)}{L(t)}$ (say, $f_2(y)$). Suppose $\frac{P_{y_2}(t)}{P(t)}$ does not equal to $\frac{L_x(t)}{L(t)}$ for any $x \in [0,\omega)$, then we denote y_2 by z_2(if $z_1$already arrives earlier for the situation of $f_1(z_1) \neq f_2(z_1)$), and write this situation as $f_1(z_2) \neq f_2(z_2)$. However, if $f_1(y_1) = f_2(y_1)$ exists but $\frac{P_{y_2}(t)}{P(t)} \neq \frac{L_x(t)}{L(t)}$ $\forall x \in [0,\omega)$, then y_2 we denote as z_1.

Similarly, for the age y_i, the fraction $\frac{P_{y_i}(t)}{P(t)}$ is matched with the fraction $\frac{L_x(t)}{L(t)}$ $\forall x \in [0,\omega)$. If there is a value of $\frac{P_{y_i}(t)}{P(t)}$ that matches with $\frac{L_x(t)}{L(t)}$ then we denote it by $f_2(y)$,otherwise we denote it by $f_2(z_i)$ (if previous z value in the order of unmatched fractions was z_{i-1} for $i = 2,3,...$). Through this procedure we will match for all the values of $y_j \in [0,\omega)$.

Let $\{y_{k_j}\}$ be the set of all j values for which corresponding matched value in $\frac{L_{y_{k_j}}(t)}{L(t)}$ exists and $\{z_{k_l}\}$ be the set of all l values for which corresponding matched value in $\frac{L_{y_{k_l}}(t)}{L(t)}$ do not exist.

Here,

$$\{y_{k_j}\} \bigcup \{z_{k_l}\} = [0,\omega),$$

and corresponding population totals

$$\Sigma_{j=1}^{\infty} P_{y_{k_j}}(t) + \Sigma_{l=1}^{\infty} P_{y_{k_l}(t)},$$

will be equal to $P(t)$, where $\Sigma_{j=1}^{\infty} P_{y_{k_j}}(t)$ is $M(t)$ formed by satisfying the SPI, and $\Sigma_{l=1}^{\infty} P_{y_{k_l}(t)}$ is $N(t)$ formed by not satisfying the SPI. Hence, the proof.

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